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Propagation into an Unstable State

Gregory Dee[1,2]

We describe propagating front solutions of the equations of motion of pattern-forming systems. We make a number of conjectures concerning the properties of such fronts in connection with pattern selection in these systems. We describe a calculation which can be used to calculate the velocity and state selected by certain types of propagating fronts. We investigate the propagating front solutions of the amplitude equation which provides a valid dynamical description of many pattern-forming systems near onset.

KEY WORDS: Propagating front; pattern selection; natural velocity; marginal stability; amplitude equation.

1. INTRODUCTION

There are many dissipative pattern-forming systems in nature. Examples of such systems include Rayleigh–Bénard convection,[1] Taylor–Couette flow,[2] directional solidification,[3] and many biological and chemical systems. These systems have a number of common features. In particular, there exists a "control" parameter, v say, which for $v < v_c$ no stable pattern states exist. For values of v just above v_c, however, stable pattern states exist. We define a pattern state to be a stationary solution of the equations of motion which is a periodic function of the spatial variables. Close to onset, i.e., for $v \gtrsim v_c$, the pattern state is dominated by a single mode. There is a dynamical description of these systems close to onset which is given in terms of a dynamical equation for the amplitude of this dominant mode. This equation is known as the amplitude equation.[4]

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[1] Institute for Theoretical Physics, University of California at Santa Barbara, Santa Barbara, California 93106.

[2] Address after September 1, 1984: Central Research and Development Department, DuPont Experimental Station, Wilmington, Delaware 19898.

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The dynamical equations which describe these systems allow for a set of possible stable pattern solutions above onset. The question arises, therefore, is there a preferred state among this set, which is selected by different initial and boundary conditions? The answer to this question is no. However, there are particular dynamical mechanisms[5] and boundary conditions[6] which do select specific states. In this paper I discuss one such dynamical mechanism which selects a particular state for a large class of pattern-forming systems. The dynamical mechanism has been termed *pattern propagation.*[5] This dynamical mechanism can occur under the following circumstances. Suppose we change the parameter v so that the system finds itself in a stationary but unstable state. The system is then perturbed locally at some point, at the origin say, and subsequently decays to a new state. This will proceed by the new stable state forming first at the point where the system was disturbed and then a front will develop which separates the stable and unstable states. This front will propagate into the unstable state and it is the properties of such a front that we address in this paper.

As a mathematical problem this phenomenon has received much attention in the past in connection with the velocity of propagating fronts for reaction diffusion equations.[7] However, it is the pattern selection properties of the phenomenon which are of interest to us in this paper. In an effort to understand the dynamical properties of these rather complex and diverse systems we put forward some conjectures concerning the pattern selection properties of propagating fronts. We also present some tractable calculational tools[5] which enable us to predict some of the properties of these fronts for this class of pattern-forming systems. Since the amplitude equation describes the dynamics of these systems close to onset, it seems appropriate to study the properties of propagating front solutions of these equations first.

In Section 2 we briefly revise some of the details of the amplitude equation. We describe the two possible propagating fronts which can occur as solutions of the amplitude equation. We make three conjectures concerning the pattern selection properties of these fronts. In Section 3 we present a calculational tool for predicting the velocity of these fronts and in some circumstances the state selected by the front. We also discuss the initial transients caused by the initial conditions. In Section 4 we present numerical results for the amplitude equation. In Section 5 we present some conclusions.

2. THE AMPLITUDE EQUATION

The state of a pattern-forming system is described by a function u, where u is in general a vector function. The dynamical equations for the system are in general nonlinear and can be written in the form

$$F[u] = 0 \tag{2.1}$$

where F is a differential or integrodifferential operator. There exists a stationary solution of Eq. (2.1) which we call a homogeneous state and for the sake of simplicity we choose this to be the state $u_s = 0$. The term *homogeneous* simply implies that there is no pattern present. A standard practice in such nonlinear problems is to perform a linear stability analysis about this state. To do this we consider perturbations of the state $u_s = 0$ which are of the form

$$\phi \sim e^{ikx + \omega t} \tag{2.2}$$

We will restrict all following discussions to one-dimensional systems. Substituting (2.2) into the equation of motion (2.1) and linearizing in ϕ, where ϕ is assumed to be small, we can obtain the dispersion relation $\omega(k, v)$. In Fig. 1 we show a plot of $\omega(k, v)$ for different values of v for a typical pattern-forming system. Since positive values of the real part of ω indicate instability, the pertinant feature of Fig. 1, which is common to the pattern-forming systems mentioned in Section 1, is that ω as a function of v first becomes positive at a finite wave vector k_c.

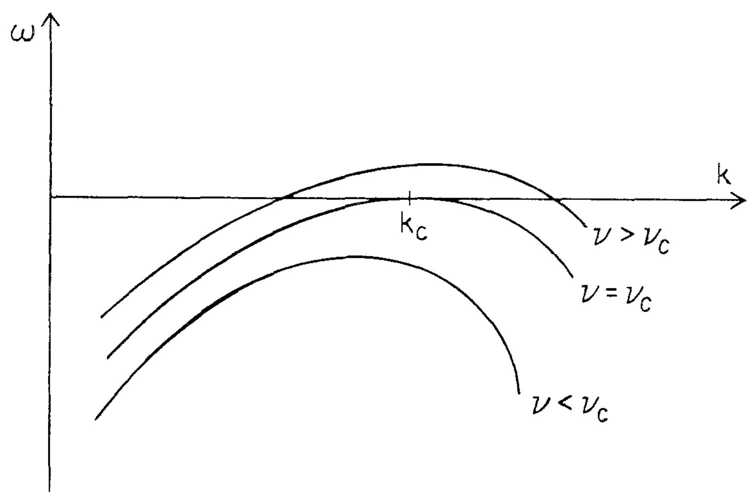

Fig. 1. A plot of the function $\omega(k, v)$ for values of v less than, equal to, and greater than v_c.

To describe the dynamics of the system close to v_c we can make use of two facts. The first of which is that for $v \gtrsim v_c$ a small band of wave vectors Δk exists for which the system is unstable. Secondly, $\omega(k, v)$ is small for wave vectors in this band Δk. This reflects the fact that the linear growth rate of these unstable modes is small. If a pattern solution exists just above onset, we expect it to have a wave vector which is close to k_c. These facts lead us to make the following ansatz for u:

$$u \cong A(x, t) e^{ik_c x} + A^*(x, t) e^{-ik_c x} \tag{2.3}$$

where A is a complex function and depends weakly on x and t. Since a clear separation of space and time scales exists, we can use the standard methods of multiple-scales[8,4] analysis to derive the equation of motion for $A(x, t)$. This equation is known as the amplitude equation[4] and is of the form

$$\frac{\partial A}{\partial t} = \partial_x^2 A + \Delta v A - A^* |A|^2 \tag{2.4}$$

The coefficients in Eq. (2.4) have been set to one but they will in general depend on the parameters of the model. However, by rescaling A, x, and t it is always possible to write the equation in the form given by (2.4).

Stationary solutions of Eq. (2.4) are of the form

$$A_s(x) = [\Delta v(1 - q^2)]^{1/2} e^{iq(\Delta v)^{1/2} x} \tag{2.5}$$

These solutions correspond to patterns with a fundamental wave number $k = k_c \pm (\Delta v)^{1/2} q$. Stationary solutions exist for all values of k for which $\omega(k, v) \geqslant 0$. Those solutions for which $q^2 > q_0^2 = 1/3$ are found to be unstable to an instability known as the Eckhaus instability. This instability is associated with long wavelength phase disturbances. A band of stable solutions exists for $q^2 < q_0^2$. This situation is illustrated in Fig. 2 where we show a plot of v versus k showing the neutral stability curve, i.e., the points in the v, k plane where $\omega(k, v) = 0$. This line is denoted by the full curve. Stable pattern states exist for those values of v and k above the dashed curve.

There are two possible quenches in the value of v which we can perform in order to produce the conditions necessary in order to form a propagating front. These two options are denoted by the labels (A) and (B) in Fig. 2. In case (A) we begin with the system at a value of $v_i < v_c$ where the homogeneous state is stable and quench to a value of $v_f > v_c$ where this state is now unstable. We then disturb the homogeneous state locally at the origin and observe the decay to stable solution. After some initial transients

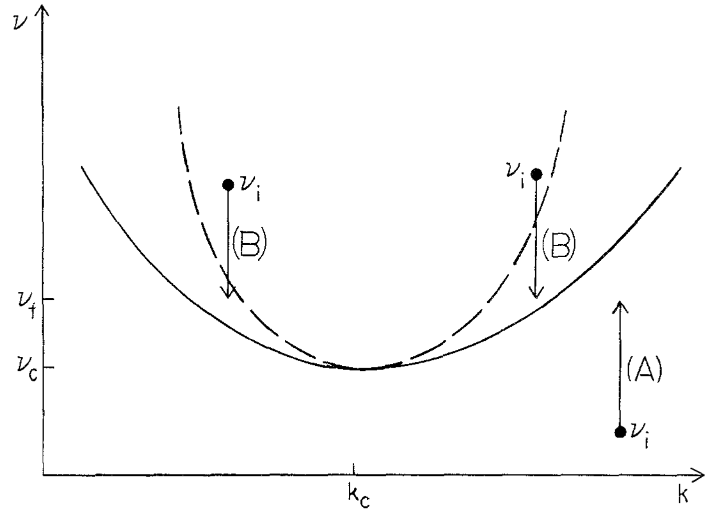

Fig. 2. A plot showing the neutral stability curve, which is denoted by the full line. Stable stationary pattern states exist above the dashed line. Also shown are the two different types of possible quenches for this system, which are labeled (A) and (B).

we will observe the formation of a propagating front which connects the homogeneous state to one of the allowed pattern states. An example of such a front for the amplitude equation is shown in Fig. 3a, where we plot the modulus of the amplitude A as a function of x and t. We next consider the quench labeled (B) in Fig. 2. In this case we start at $v_i > v_c$ where the system is in a stable stationary pattern state. Then v is changed to a value $v_f > v_c$. There exists a stationary solution of the equations of motion at this value of v and k. This state, however, is unstable to long wavelength phase disturbances. The initial state will relax to the stationary solution at v_f and this values of k since this state is stable against amplitude perturbations. We can now introduce a small local disturbance at the origin and observe the subsequent decay of the pattern state to one of the stable pattern states at this value of v. As in case (A) a propagating front will develop but this front will connect two pattern states. Such a front is shown in Fig. 3b for the amplitude equation. We observe that the front exhibits quite complicated behavior in this case but that the mean position of the interface between the two patterns seems to move at a constant velocity. This behavior at the interface results from the fact that the two patterns have different spatial periodicities. Therefore in order for the stable pattern to propagate into the unstable pattern, the phase of the system must change discontinuously at the front. At such points the phase changes by $\pm 2\pi$ and the amplitude of the pattern at the interface goes to zero. This results in the dynamical behavior observed in Fig. 3b for the amplitude equation.

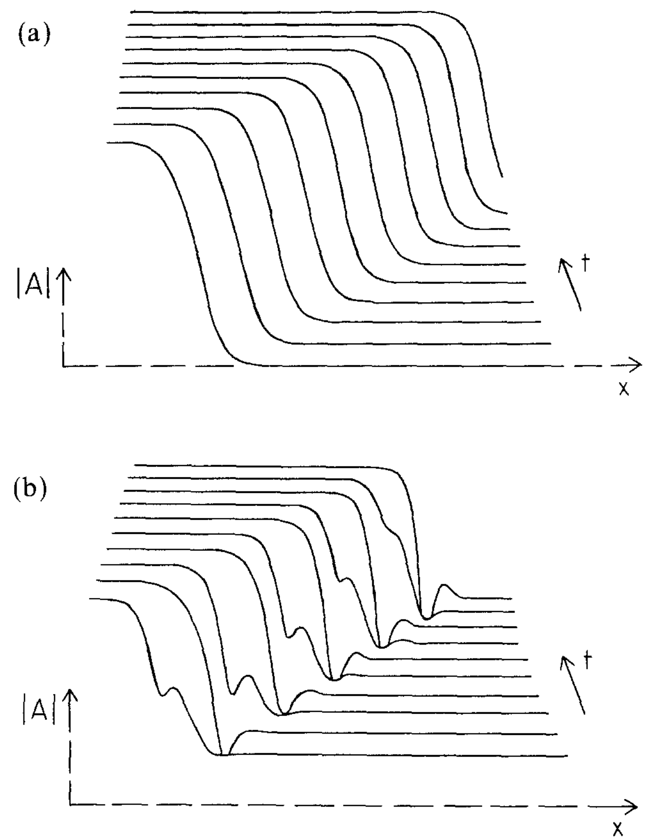

Fig. 3. (a) A propagating front of type (A) for the amplitude equation. We plot the modulus of the complex amplitude as a function of x for various equally spaced times. (b) A front of type (B) for the amplitude equation at equally spaced times, where again the modulus of the amplitude is plotted versus the x coordinate.

These two types of propagating-front solutions will exist away from onset and the quantitative features may differ from system to system. We would like to predict the following properties of these propagating fronts: (a) What is the velocity of the front (b) what is the state selected in the wake of such a front? We will answer these questions here for the amplitude equation. Before we do this, however, we will make a number of conjectures about these types of propagating-front solutions in general.

(i) There exists a natural propagation speed c^* such that, if an observer moves with a velocity c, the subsequent evolution is seen to approach the unstable state as $t \to \infty$ if $c > c^*$, but not to do so for $c < c^*$.

(ii) There exists a naturally selected pattern k^* which will be seen as $t \to \infty$ by an observer who is not moving. This observer, however, must be located at large enough x so that the pattern he sees is not effected by transients that may persist near the initial perturbation.

(iii) The naturally propagating mode (c^*, k^*) is marginally stable against localized perturbations, that is, against perturbations which decay sufficiently fast as $x \to \infty$. This marginal stability conjecture is equivalent to the statement that c^* is the slowest stable propagating speed provided one excludes from consideration certain slowly propagating solutions which are, for topological reasons, inaccessible from the stated initial conditions. To demonstrate that this last conjecture is indeed true persents a difficult task. One has to demonstrate the existence of all possible propagating solutions and then perform a stability analysis of these solutions.

3. THE PROPERTIES OF THE PROPAGATION FRONT

In what follows we present a method for computing c^* and k^* by means of a linear stability analysis of the unstable state in a moving frame. Before this, however, we consider the nature of the initial transient evolution in the vicinity of the initial perturbation. Since the perturbation is small we can linearize the equations of motion. The solution of the linearized equations is of the form

$$u(x, t) = \frac{1}{(2\pi)^{1/2}} \int dk\, \tilde{u}(k)\, e^{ikx + \omega(k)t} \tag{3.1}$$

where $\omega(k)$ is the dispersion relation which can be obtained as described in Section 2. At late times, i.e., as $t \to \infty$, the integral (3.1) can be approximated by performing a steepest decent calculation to give

$$u(x, t) \cong \hat{u}(\bar{k})\, e^{i\bar{k}x + \omega(\bar{k})t} \tag{3.2}$$

where $\bar{k}$ is defined to be such that

$$\left.\frac{\partial \omega(k)}{\partial k}\right|_{k=\bar{k}} = 0 \tag{3.3}$$

We expect the region occupied by the initial transients to be dominated by the mode with wave number $\bar{k}$. This wave number is sometimes termed the *fastest growing mode* since $\omega(k)$ has a maximum at this wave number. Crucial to the above argument is the fact the initial perturbation be small.

We now turn our attention to computing c^* and k^* using an approach similar to that described above but in the spirit of conjecture 1.

We first make a transformation to a frame of reference moving at a velocity c in the positive x direction, say. Then the solution of the linearized equations in this frame of reference can be written in the form

$$u(x, t) \cong \frac{1}{(2\pi)^{1/2}} \int dk\, \hat{u}(k)\, e^{-ikx + \omega(k)t + ickt} \tag{3.4}$$

In the case where u is a vector function we can perform a transformation on u so that the linearized operator is diagonal. In Eq. (3.4) k and $\omega(k)$ are now complex. However, as before, we can compute the point of stationary phase from the equation

$$\frac{\partial}{\partial k} \Omega(k, c) \bigg|_{k_m} = 0 \qquad \text{where} \quad \Omega = \omega(k) + ick \tag{3.5}$$

We then solve for k_m, which depends on c. Returning to conjecture 1 we note that if c is equal to c^* we expect that $\mathrm{Re}(\Omega)$ will be zero. This simply implies that we are in a frame of reference moving with the front. This does not exclude the fact that there may be some local time periodic behavior associated with the front as in case (B) described in Section 2. Therefore, to compute c^* for a system where $\omega(k, v)$ is known we simply compute k_m using (3.5) and then use the following equation to compute c^*, i.e.,

$$\mathrm{Re}(\Omega(k_m, c^*)) = 0 \tag{3.6}$$

It should be pointed out that the above calculations may not always yield the correct values of c^* as nonlinear aspcts of the motion of the front may be important. However, in most cases connected with pattern-forming systems the above calculation has proved successful. We should also state that the three conjectures are not invalidated in the cases where is does not succeed.

To calculate the selected state one has to make a clear distinction between cases (A) and (B). In case (A) a pattern is forming in the presence of a homogeneous state. We can therefore use details of the linear stability analysis presented above to predict the final state[5] as follows. There are nodes formed as the leading edge of the front advances into the unstable state. In the frame of reference moving with the velocity c^*, therefore, there is a flux of nodes in the $-x$ direction at the front or at a point in the stable pattern in the wake of the front. At the front this flux of nodes is given by $\mathrm{Im}[\Omega(k_m, c^*)]$. In the bulk this flux is equal to k^*c^*, where k^* is the wave vector of the bulk pattern in the wake of the front. Therefore equating these two expressions we obtain the result

$$k^* = \mathrm{Im}[\Omega(k_m, c^*)]/c^* \tag{3.7}$$

For case (B)[10] one cannot predict exactly the final stable pattern from details of the linear stability analysis. This results from the fact that the initial and final states are connected in time by nonlinear events associated with the discontinuous change in the phase of the system which must occur at the front in order for it to advance. These points at which the phase changes discontinuously by 2π are termed *phase slip centers*[11] (P.S.C.'s). Clearly, once the front has formed and moved well away from the region occupied by the initial transients, if we know the positions of the P.S.C.'s we can calculate the wave vector of the final state to be

$$k_f = k_i \pm k_s \tag{3.8}$$

where k_s is $2\pi/\lambda_s$ and λ_s is the mean separation between the phase slip centers. The plus an minus sign in Eq. (3.8) allows for the fact that the phase may increase or decrease by 2π at each P.S.C. We define λ_s to be the mean separation between the P.S.C.'s because a complex but repeated pattern may result owing to the nonlinear nature of these events. This fact will become apparent when we consider the numerical results for the amplitude equations.

If the nonlinear nature of these P.S.C.'s could be ignored, then one would predict that the P.S.C.'s would occur with a periodicity given by $\lambda_m = 2\pi/\text{Re}(k_m)$. This would lead to a prediction of the final state wavevector being

$$k_f = k_i \pm \text{Re}(k_m) \tag{3.9}$$

It will be interesting to check Eq. (3.9) with the results of the numerical simulations for the amplitude equation.

In Fig. 4 we show the velocity c^* as calculated using Eqs. (3.5) and (3.6) for the amplitude equations. Case (A) corresponds to $q_i = 1$ and we calculate c^* to be 2. The other values of q, i.e., $q_0 < q < 1$, correspond to case (B). We see that c^* approaches zero at q_0. In Fig. 5 we plot the bulk-selected wave vector for the amplitude equation. For case (A) we predict using (3.7) $q_f = 0$. Also shown for case (B) is the prediction based on Eq. (3.9), which we do not expect to be exactly correct.

From the above discussion we can draw some obvious conclusions concerning the pattern state selected in a system which has evolved via the formation of these fronts. Suppose the fronts are initiated at equally spaced points. If these points are close together then it is obvious that the subsequent evolution will be dominated by the initial transients described above, and hence the wave vector selected will be close to $\bar{k}$ where $\bar{k}$ is the fastest growing mode. If the initiation points are very far apart, then the fronts will form and the evolution of the system will be dominated by these propagating solutions. Therefore the final state selected in such a system

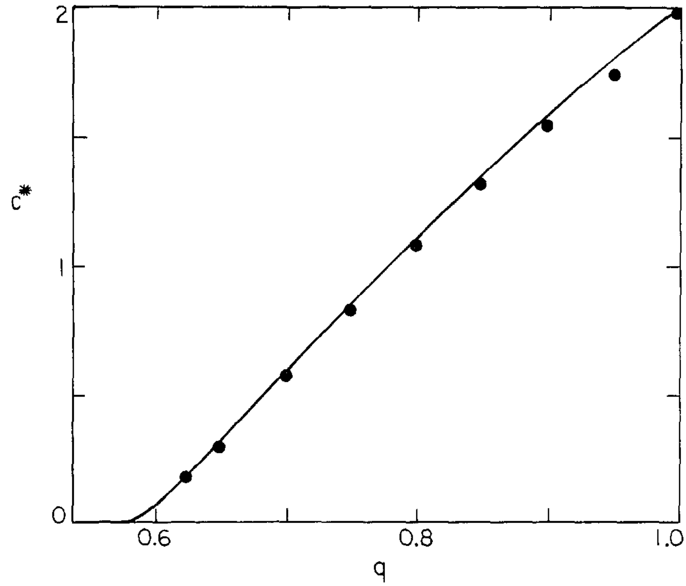

Fig. 4. A plot of the natural velocity c^* versus the value of q for the initial state. The dots indicate the results of the numerical simulation.

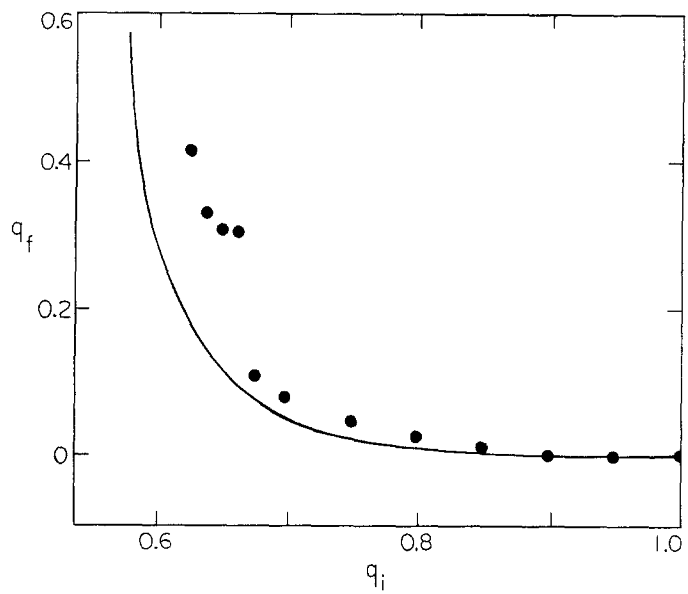

Fig. 5. A plot of the quantity q_f selected by the propagating front for different values of q_i. The dots indicate numerical observations.

will be close to k^*. At intermediate values of the initial spacing we expect the final state to be between $\bar{k}$ and k^*. For the amplitude equations we see from the above analysis of fronts of type (A) that $\bar{k}$ and k^* are the same. However away from onset $\bar{k}(v)$ and $k^*(v)$ are different[5] and the above argument suggests that if the evolution of the system takes place via these propagating fronts then the final state will be bounded by $\bar{k}$ and k^*.

4. NUMERICAL RESULTS

To test the calculations of the last section, we performed numerical simulations on propagating-front solutions of the amplitude equation, of the types (A) and (B) as described in the previous sections. To perform the numerical simulation we define new variables u_1 and u_2 such that

$$A = u_1 + iu_2 \tag{4.1}$$

Then u_1 and u_2 satisfy the following dynamical equations:

$$\frac{\partial u_1}{\partial t} = \partial_x^2 u_1 + u_1(1 - u_1^2 - u_2^2) \tag{4.2a}$$

$$\frac{\partial u_2}{\partial t} = \partial_x^2 u_2 + u_2(1 - u_1^2 - u_2^2) \tag{4.2b}$$

where we have set $\Delta v = 1$ for convenience. We used an implicit scheme to integrate Eqs. (4.2). Initially the whole system was occupied by the unstable state which for case (A) was $u_1 = 0$, $u_2 = 0$ and for case (B) was

$$u_1 = (1 - q_i^2)^{1/2} \cos(q_i x) \qquad \text{and} \qquad u_2 = (1 - q_i^2)^{1/2} \sin(q_i x)$$

We then perturbed the initial state at the origin. After some initial transients a front developed and propagated into the unstable state. For case (A), as illustrated in Fig. 3a we simply tracked the position of the interface and computed the velocity. We also monitored the state in the wake of the front. In case (B), as illustrated by Fig. 3b, we recorded the positions of the phase slip centers, i.e., the points where A was zero, and hence computed the velocity of the front. As in case (A) we also monitored the state in the wake of the front.

The numerical results for the velocity are shown in Fig. 4. The dots indicate the numerical results. Better agreement can be obtained between the theoretical predictions and the numerical measurements by more accurate numerical procedures, i.e., by decreasing the grid sizes for the space and time variables for instance. In Fig. 5, we show the numerical results for the state selected in the wake of the front. The pattern states are specified by the value of q, where the states for which $q_0 < q < 1$ are unstable and the states for which $0 < q < q_0$ are stable. For case (A) the theory and the numerical results agree exactly. For case (B), as expected, the agreement between Eq. (3.9) and the numerical results decreases as the nonlinear nature of the initial condition increases, i.e., as q_i decreases from $q_i = 1$. At $q_i = 0.6625$ a discontinuity seems to occur in the data. This corresponds to a change in the behavior of the front as a function of time. For values of $q_i > 0.6625$ the P.S.C.'s are equally spaced. For $q_i \lesssim 0.6625$ a more complicated pattern of P.S.C.'s results simply reflecting the nonlinear nature of this front.[11] At $q_i \cong 0.6325$ a second discontinuity occurs corresponding to a further change in the pattern of P.S.C.'s at the front. Clearly more data are needed to establish the exact nature of this curve. It is clear from these numerical results, however, that (a) all the selected states lie in the set of allowed stable solutions and (b) that gaps exist in the band of allowed states which are not accessible by a system evolving via the mechanism of propagating fronts.

5. CONCLUSIONS

We have described two different types of propagating front solutions of the amplitude equation. The qualitative features of these solutions are expected to persist away from onset[5] and it will be of interest to investigate such solutions using the techniques outlined in Section 3. Using these analytical methods one can calculate the velocity of both types of fronts and in the case of fronts of type (A) one can also predict the state selected in the wake of the front.

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